ELC 504
Module 1: Introduction to Digital Communication System and Probability Theory

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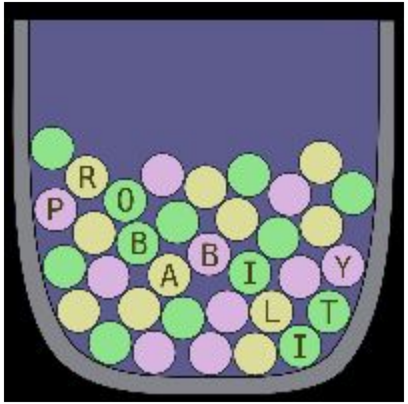
Content

- ❑ Introduction to Digital Communication System
- ❑ Probability Theory
- ❑ Probability Distributions



A typical Communication System

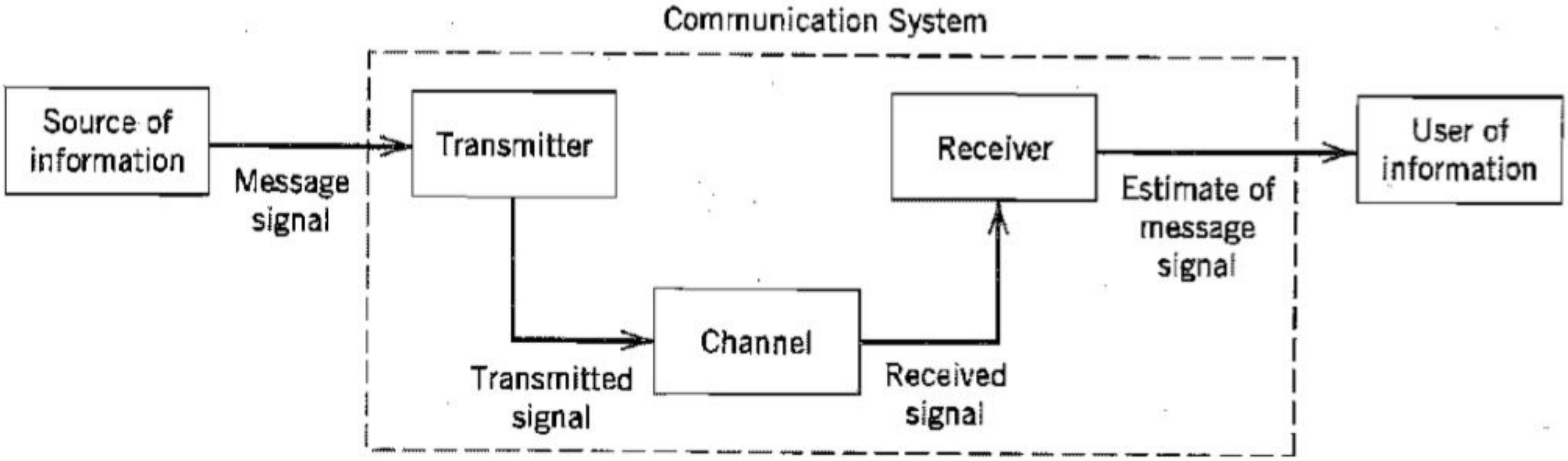


FIGURE 1 Elements of a communication system.

A Digital Communication System

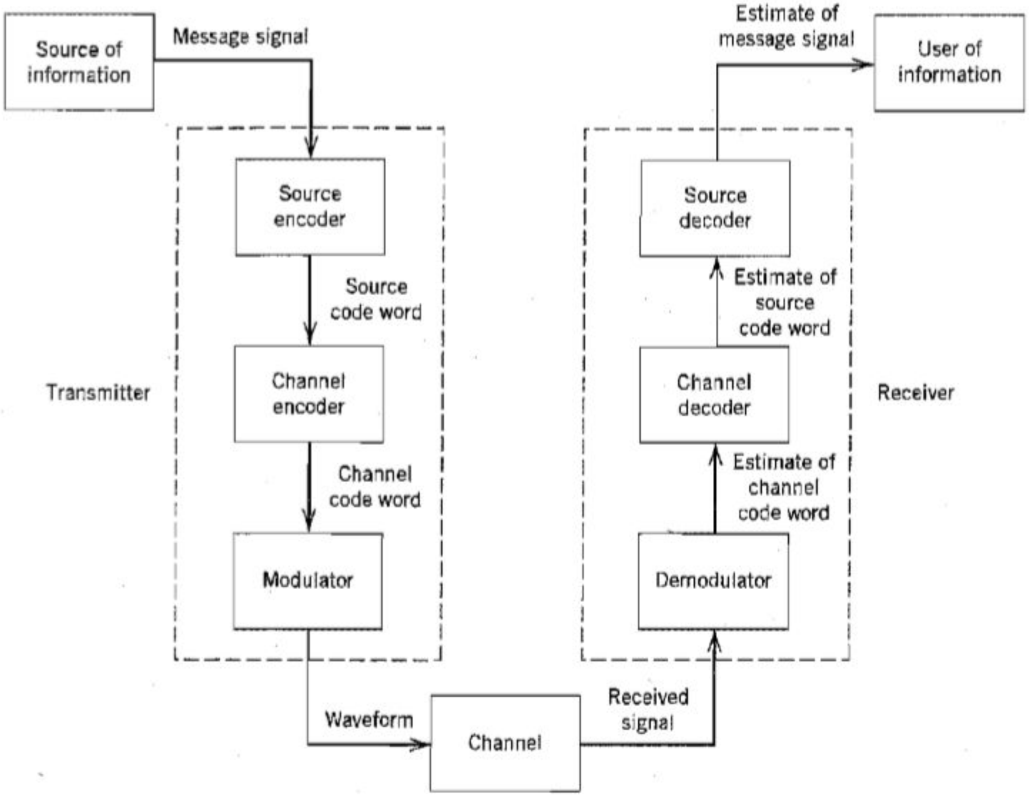
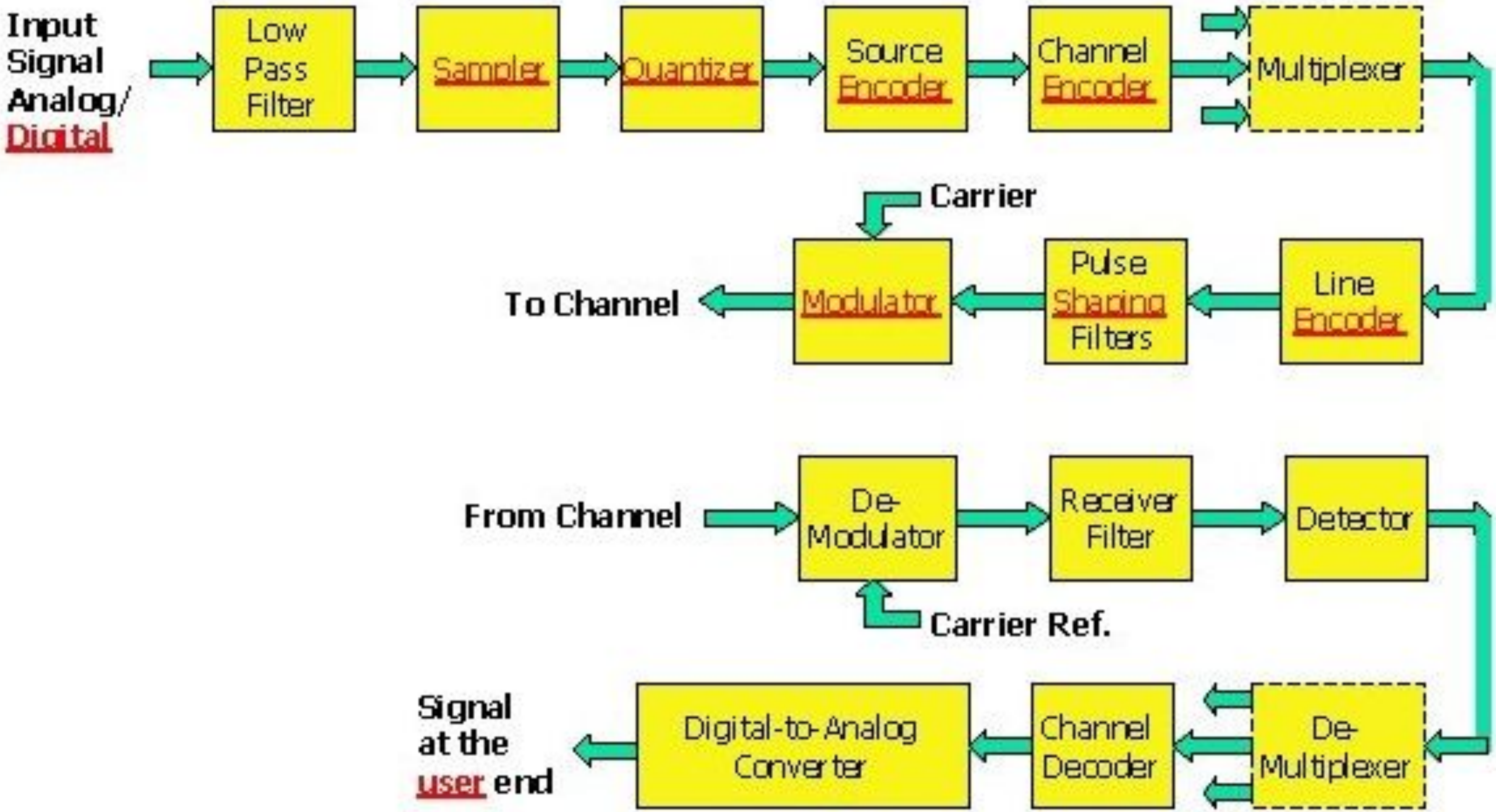


FIGURE 9 Block diagram of digital communication system.


Input Signal Analog/ Digital
Low Pass Filter
Sampler
Quantizer
Source Encoder
Channel Encoder
Multiplexer
Carrier
To Channel
Modulator
Pulse Shaping Filters
Line Encoder
From Channel
De-Modulator
Receiver Filter
Detector
Carrier Ref.
Signal at the user end
Digital-to-Analog Converter
Channel Decoder
De-Multiplexer

Principal feature of digital over analog communication system

The principal feature of a digital communication system (DCS) is that during a finite interval of time, it sends a waveform from a finite set of possible waveforms, in contrast to an analog communication system, which sends a waveform from an infinite variety of waveform shapes with theoretically infinite resolution. In a DCS, the objective at the receiver is *not* to reproduce a transmitted waveform with precision; instead, the objective is to determine from a noise-perturbed signal which waveform from the finite set of waveforms was sent by the transmitter.

Advantages of Digital Transmission

- Reliable communication; less sensitivity to changes in environmental conditions (temperature, etc.)
- Easy multiplexing
- Easy signaling
  - Hook status, address digits, call progress information
- Voice and data integration
- Easy processing like encryption and compression
- Easy system performance monitoring
  - QOS monitoring
- Integration of transmission and switching
- Signal regeneration, operation at low SNR, superior performance
- Integration of services leading to ISDN

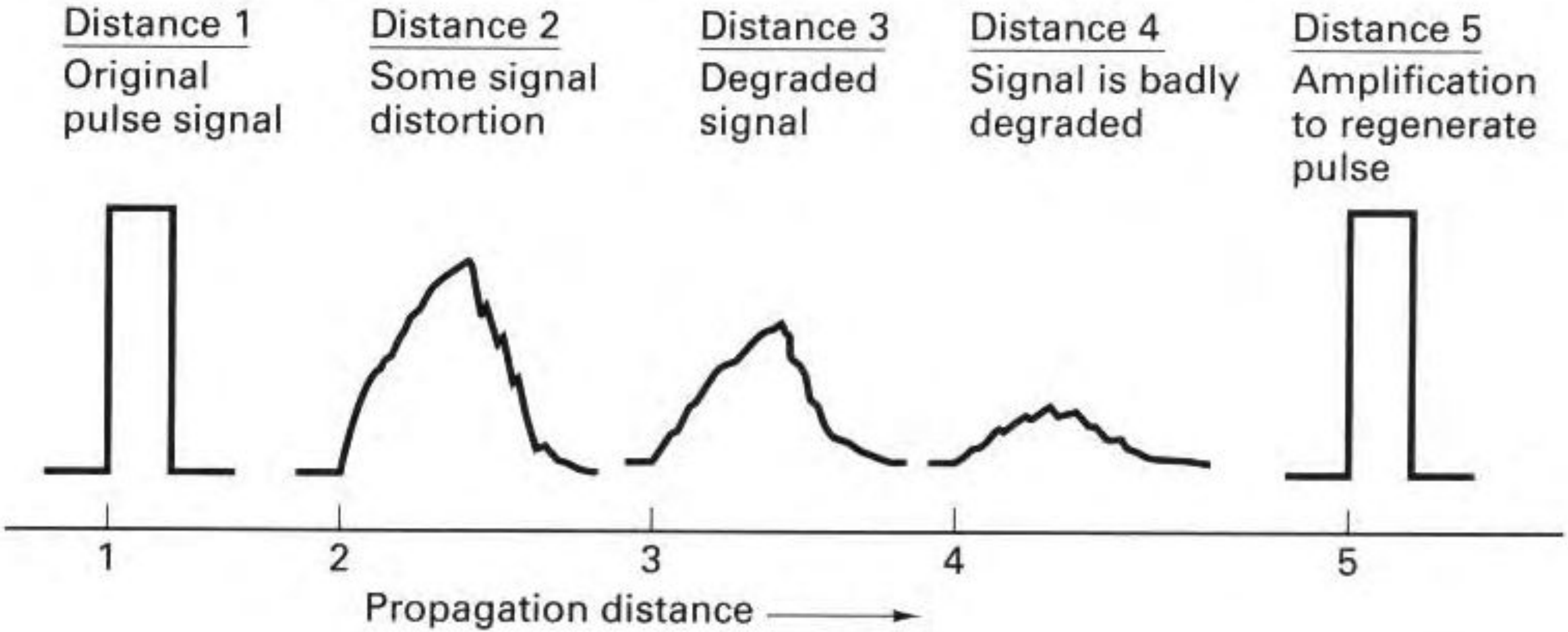


Figure 1.1 Pulse degradation and regeneration.

Disadvantages of Digital Transmission

- Increased bandwidth
  - 64 KB for a 4 KHz channel, without compression (However, less with compression)
- Need for precision timing
  - Bit, character, frame synchronization needed
- Analogue to Digital and Digital to Analogue conversions
  - Very often non-linear ADC and DAC used, some performance degradation
- Higher complexity

Significance of Digitization

- ❖ Digitization is the process of converting analog signals or information of any form into a digital format that can be understood by computer systems or electronic devices

- ❖ The term is used when converting information, like text, images or voices and sounds into binary code

Significance of Digitization…..

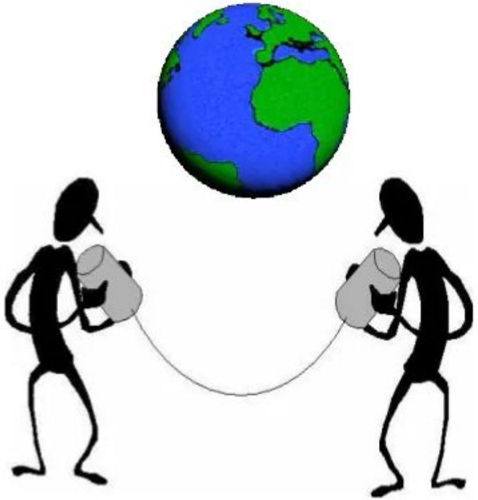
- Digitized information is easier to store, access and transmit, and digitization is used by a number of consumer electronics devices

- One of the most important qualities of information in digital form is that by its very nature, it is not fixed in the way that texts are printed on paper

Significance of Digitization…..

- ❖ Flexibility is one of the chief assets of digital information.
- ❖ It is easy to edit , to reformat, and to commit print in a variety of iterations without the effort required to produce hard copy.
- ❖ Furthermore, we can create an endless number of identical copies from a digital file, because the file does not decay by virtue of copying.

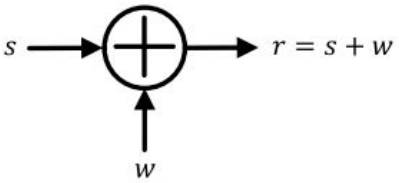
Communication Channel

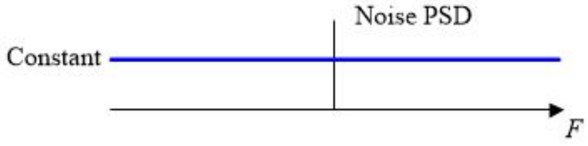


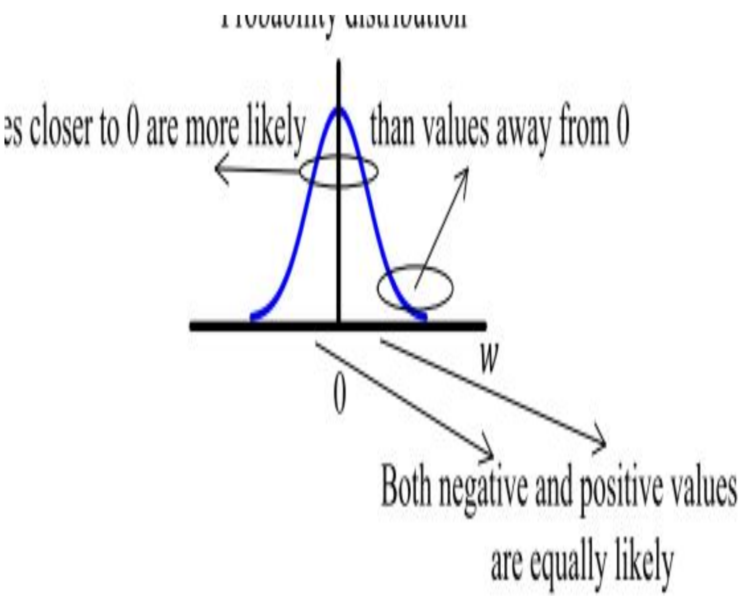
- A communication channel is used to convey an information signal.
- A channel has a certain capacity for transmitting information, often measured by its bandwidth in Hz or its data rate in bits per second.
- Mathematical models of the channel can be made to describe how the input (the transmitted signal) is mapped to output (the received signal).

Significance of AWGN

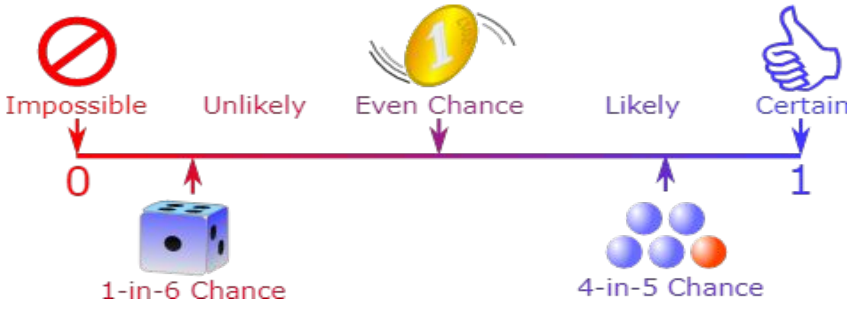
AWGN- Additive White Gaussian Noise







Probability Theory



-Probability theory provides mathematical rules for assigning probabilities to outcome of random experiment e.g. Coin tossing, packet arrival, noise voltage

-Probability theory is applied to situations where uncertainty exists

-Engineers apply the theories of Probability and Random Processes to those repeating situations in nature where

1. *We can roughly predict what may happen*
2. *We cannot exactly determine what may happen*

-Whenever we cannot exactly predict an occurrence, we say that such an occurrence is random

Why study Probability Theory in Digital Communication?

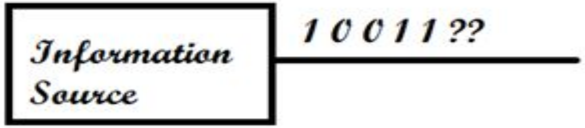
- The main objective of a communication system is the transfer of information over a channel
- All useful message signal appear random, i.e. the receiver does not know, a prior, which of the possible message waveform will be transmitted.
- Also nature of noise is random, that arises due to electrical signals
- Randomness affects the performance of communication System.
- When message signal is transmitted through a channel (wired/wireless) it gets corrupted by noise.
- To recover the message signal, we use probability theory for estimation

- Symbols transmitter randomly
- Transmit average power
- Probability of Bit error

Important definitions in Probability

- Random Experiment

An experiment is called as random experiment if the outcome of the experiment cannot be predicted precisely



- Sample Space

The sample space S is defined as a collection of all possible, separately identified outcomes of a random experiment.

E.g. 1. sample space of tossing a coin

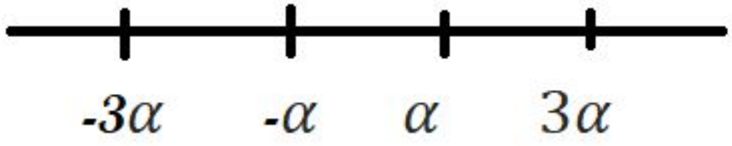
{H,T}

2. Rolling a die

{1,2,3,4,5,6}

Example: M-ary PAM

Let M= 4 i.e. there are four symbols

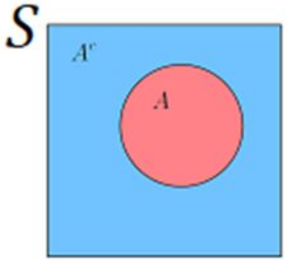


Sample Space S={-3α.-α,α,+3α}

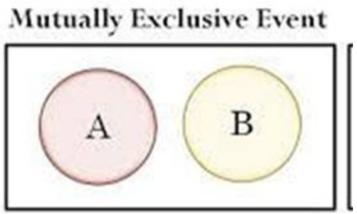
Important definitions in Probability

- Event:- Event is defined as a subset of sample space or outcomes meeting some specification

E.g. A={-3α.α} i.e. A ⊂ S



- ❖ Complement of an event
- ❖ Union of an Event A U B
- ❖ Intersection of Event A ∩ B
- ❖ Null Event = Φ
- ❖ Mutually Exclusive Event i.e. A ∩ B = Φ
- ❖ Mutually Independent Event



$P(A \cup B)=P(A)+P(B)-P(A \cap B)$

E.g. $S=\{-3\alpha.-\alpha,\alpha,+3\alpha\}$ with probabilities {1/8.1/8,1/4,1/2}

$P(S)=?$

$A=\{-3\alpha.\alpha,\}$

$P(A)=?$

$B=\{\alpha.3\alpha,\}$

$P(B)=?$

$(A \cup B)=\{-3\alpha.\alpha,\} \cup \{\alpha.3\alpha,\}=\{-3\alpha.-\alpha,+3\alpha\}$

$P(A \cup B)=?$

$(A \cap B)=\{-3\alpha.\alpha,\} \cap \{\alpha.3\alpha,\}=\{\alpha\}$

$P(A \cap B)=?$

Hence proved

Definition of Probability

I. Relative Frequency Definition

- Suppose that one of the possible outcome of an experiment is called A and that when the experiment is repeated N times the outcome A occurs N_A The relative frequency of occurrence of A is $\frac{N_A}{N}$, and this ratio $\frac{N_A}{N}$ is not predictable unless N is very large.
- This limiting value of the relative frequency of occurrence is called the probability of outcome A written so that $P(A)$

$$P(A) = \lim_{n\to\infty} \frac{N_A}{N}$$

Definition of Probability

II. Classical Definition

$$P(A) = \frac{number\ of\ possible\ favorable\ outcomes}{total\ number\ of\ possible\ equally\ likely\ outcome}$$

1. $0 \leq P \leq 1$ (Probability is a positive number)

2. $P = 0$ (if an event is not possible)

3. $P = 1$ (if it an certain (sure)event)

Joint Probability of related and Independent Events (Bayes' Theorem)

- Suppose we perform two experiments A and B with outcomes $A_1, A_2 \ldots \ldots ..$ and $B_1\ B_2 \ldots \ldots \ldots$
- The probability of the joint occurrence of, say A_j and B_k is written $P(A_j\ and B_k)$ or more simply $P(A_j\ , B_k)$
- It may be that the probability of event B_k depends on whether A_j does indeed occur.
- Here, then we have a situation in which the outcome of the second experiment is conditional on the outcome of the first experiment.
- The probability of the outcome B_k , given that A_j is known to have occurred, is called the conditional probability and written $P(B_k/A_j\ )$.

Suppose that we perform N times (N a very large number) the experiment of determining which pair of outcomes of experiment A and B occur jointly.

Let N_j be the number of times A_j occurs with or without B_k ,

N_k the number of times B_k occurs with or without A_j

And N_{jk} the number of times of joint occurrence .

Then,

$$P\left({}^{B_k}/_{A_j}\right) = \frac{N_{jk}}{N_j} = \frac{N_{jk}/_N}{N_j/_N} = \frac{P(A_j,B_k)}{P(A_j)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Similarly we have [since $N_{jk}= N_{kj}$]

$$P\left({}^{A_j}/_{B_k}\right) = \frac{N_{kj}}{N_k} = \frac{N_{kj}/_N}{N_k/_N} = \frac{P(A_j,B_k)}{P(B_k)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From 1 & 2 we have

$$P(A_j, B_k) = P(A_j)\, P\left({}^{B_k}/_{A_j}\right) = P(B_k) P\left({}^{A_j}/_{B_k}\right)$$

$$\boxed{P\left({}^{A_j}/_{B_k}\right) = \frac{P(A_j)}{P(B_k)}\, P\left({}^{B_k}/_{A_j}\right)}$$

This is Bayes theorem

Statistical Independence

Event A and B are said to be independent events if the outcome of event B has nothing to do with outcome of event A and vice versa

The Conditional probability is then

$$P(A_j, B_k) = P(A_j)P(B_k)$$

Random variables

- A random variable may be
  - ❑ Discrete
  - ❑ Continuous
- Cumulative Distribution Function (CDF)
- Probability Density Function (PDF)

The gaussian probability density function is defined as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$$

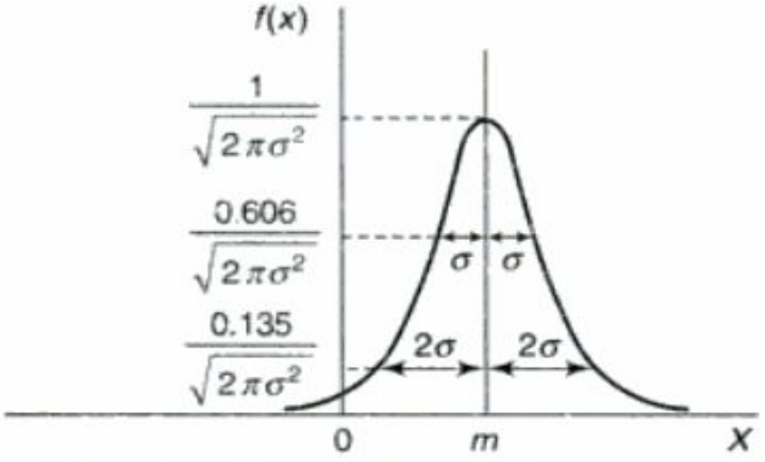


Fig. The gaussian density function.

Average Value of a Random variable

Consider now that we have the values and their associated probabilities of a discrete random variable. The possible numerical values of the random variable X are $x_1, x_2, x_3, \ldots$, with probabilities of occurrence $P(x_1), P(x_2), P(x_3) \ldots$. As the number of measurements N of X becomes very large, we would expect that we would find the outcome $X = x_1$ would occur $NP(x_1)$ times, the outcome $X = x_2$ would occur $NP(x_2)$ times, etc. Hence the arithmetic sum of all the N measurements would be

$$x_1P(x_1)N + x_2P(x_2)N + \cdots = N\sum_i x_i P(x_i)$$

The *mean* or *average value* of all these measurements and hence the average value of the random variable is calculated by dividing the sum in Eq. ( ) by the number of measurements N. The mean of a random variable X is also called the *expectation* of X and is represented either by the notation $\bar{X}$ or by $E(X)$. We shall use these notations interchangeably. Thus, using m to represent the value of the average or expectation of X, we have, from Eq.

$$\bar{X} \equiv E(X) = m = \sum_i x_i P(x_i)$$

Variance of a random variable

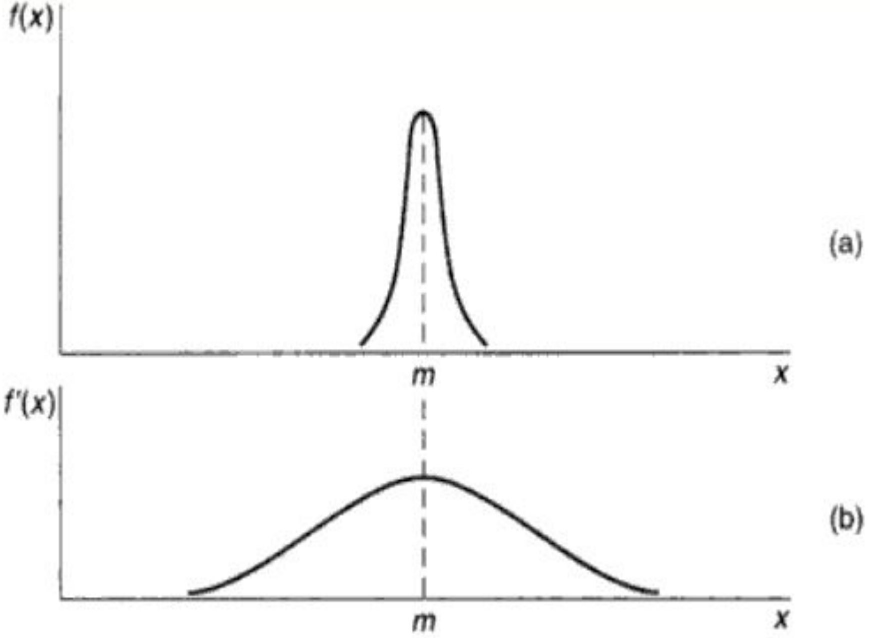


Fig. Two probability density functions corresponding to random variables with different variances.

Variance of a random variable

In Fig. are shown two probability density functions $f(x)$ and $f'(x)$ for two random variables X and X'. As a matter of simplicity we have drawn them of the same general form and have drawn them symmetrically about a common average value m. But these features are not essential to the ensuing discussion. Rather, the important point is that $f(x)$ is *narrower* than is $f'(x)$. Suppose, then, that experimental determinations were made of X and X' yielding numerical outcomes x and x'. We would surely find that, on the average, x would be closer to m than x' would be to m'. Thus in comparing X and X', we find that the outcomes of X have a higher probability of occurring in a smaller range. In other words, if a number of determinations were made of X and X', we would expect to find that the outcomes of X would *cluster* more closely around m than would be the case for X'

It is convenient to have a number which serves as a measure of the "width" of a probability density function. a more useful measure is the square root of the average value of $(X - m)^2$, that is, of the second moment of $X - m$. This second moment is represented by the symbol σ^2 and is called the *variance* of the random variable.

$$\sigma^2 \equiv E[(X - m)^2] = \int_{-\infty}^{\infty} (x - m)^2 f(x)\, dx$$

Writing $(x - m)^2 = x^2 - 2mx + m^2$ in the integral of Eq. (2.55) and integrating term by term, we find

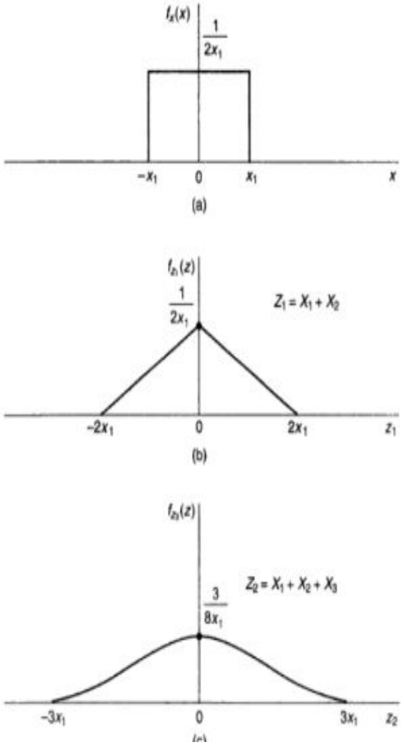
$$\sigma^2 = E(X^2) - 2m^2 + m^2$$
$$= E(X^2) - m^2$$

The quantity σ itself is called the standard deviation and is the *root mean square* (rms) value of $(X - m)$. If the average value $m = 0$, then

$$\sigma^2 = E(X^2)$$

Central Limit Theorem

It indicates that the probability density of a sum on N independent random variable tends to approach a Gaussian density as the number N increases.
The mean and variance of the Gaussian density are respectively the sum of the means and the sum of the variances of the N-independent random variable. The theorem applies even when (with a few special exceptions) the individual random variables are not Gaussian. In addition, the theorem applies in certain special cases when the individual random variables are not independent.



What is a probability distribution?

random variable is a variable whose value is the outcome of a random event.

A **probability distribution** is a list of all of the possible outcomes of a random variable along with their corresponding probability values.

**Probability Mass Function

**Probability Density Function

Various PDFs

- A random variable has a defined set of values with different probabilities.
- Random variables

Discrete

finite number of outcomes

Eg: Dead/alive, pass/fail, dice, counts etc

Continuous

infinite possibilities of values

Eg: height, weight, speedometer, real numbers etc

Distribution

- Frequency Distribution: It is a listing of observed / actual frequencies of all the outcomes of an experiment that actually occurred when experiment was done.
- Probability Distribution: It is a listing of the probabilities of all the possible outcomes that could occur if the experiment was done.
  - It can be described as:
    - A diagram (Probability Tree)
    - A table
    - A mathematical formula

DEFINITION

- A probability density function (PDF) is a function that describes the relative likelihood for this random variable to take on a given value.
- It is given by the integral of the variable's density over that range.
- It can be represented by the area under the density function but above the horizontal axis and between the lowest and greatest values of the range.

PROPERTIES

$$(i)\ f_X(x) \geq 0$$

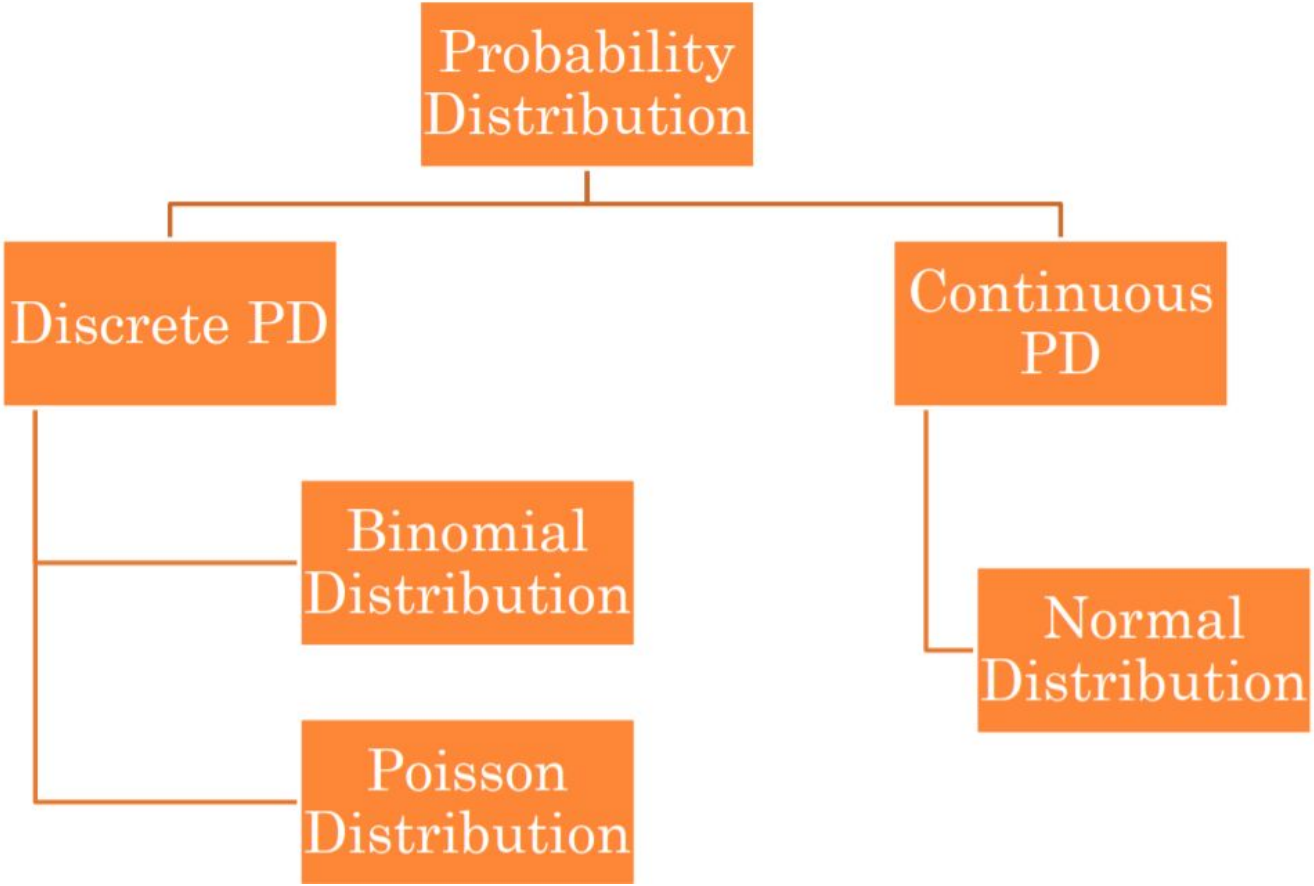
$$(ii)\ \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$(iii)\ P[a \leq X \leq b] = \int_{a}^{b} f_X(x)\,dx$$

(iv) *It* is the derivative of CDF of

a continuous random variable.

TYPES OF PROBABILITY DISTRIBUTION



BINOMIAL DISTRIBUTION

- There are certain phenomena in nature which can be identified as Bernoulli's processes, in which:
  - There is a fixed number of n trials carried out
  - Each trial has only two possible outcomes say success or failure, true or false etc.
  - Probability of occurrence of any outcome remains same over successive trials
  - Trials are statistically independent
- **Binomial distribution** is a discrete PD which expresses the probability of one set of alternatives – success (p) and failure (q)
  - **$P(X = x) = {}^{n}_{r}C\, p^{r}\, q^{n-r}$ (Prob. Of r successes in n trials)**
    - n = no. of trials undertaken
    - r = no. of successes desired
    - p = probability of success
    - q = probability of failure
- Mean of BD: $\mu = np$
- Standard Deviation of BD: $\sigma = \sqrt{npq}$

Poisson Distribution

- When there is a large number of trials, but a small probability of success, binomial calculation becomes impractical
- If λ = mean no. of occurrences of an event per unit interval of time/space, then probability that it will occur exactly 'x' times is given by
  - **P(x) =** $\frac{\lambda^x e^{-\lambda}}{x!}$ where e is napier constant & e = 2.7182

- Mean of PD is $\lambda = np$
- Standard Deviation of PD is $\sqrt{\lambda} = \sqrt{np}$

Rayleigh Distribution

- Let R be continuous random variable such that

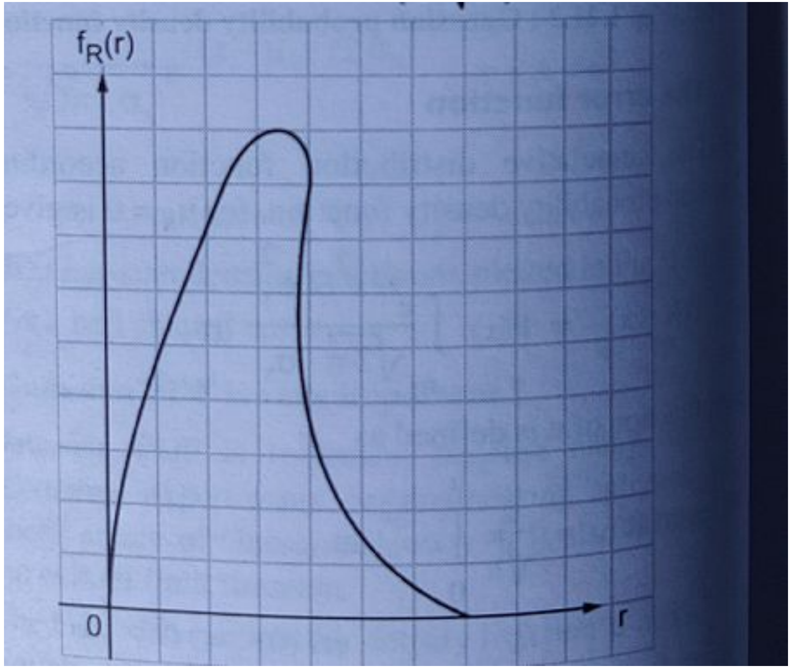
$$R = \sqrt{X^2 + Y^2}$$

where, X and Y both are Gaussian random variables.

- As X and Y are Gaussian random variables, $\mu_x = \mu_y = 0$ = mean and $\sigma_x^2 = \sigma_y^2 = \sigma^2$.

- Then the Rayleigh density is defined by,

$$f_R(\gamma) = \begin{cases} \dfrac{\gamma}{\sigma^2} e^{-\gamma^2/2\sigma^2} & , \gamma \geq 0 \\ 0 & , \gamma < 0 \end{cases}$$



Mean of Rayleigh PDF

$$\bar{R} = \sqrt{\pi/2} \cdot \sigma$$

Variance of Rayleigh PDF

$$\sigma_r^2 = (2 - \pi/2)\,\sigma^2$$

Rician Distribution

- It is important as it is useful in analysis of carrier signal passing through noisy channel.
- Rician density function is given as,

$$f_X(x) = \frac{x}{\sigma^2} e^{\frac{x^2 + a^2}{2\sigma^2}} I_o\left(\frac{ax}{\sigma^2}\right)$$

Normalized Rician PDF is, $f_X(x) = x e^{\frac{x^2 + a^2}{2}} I_o(ax)$

I_o = modified Bessel function of first kind, zeroth order

σ = standard deviation

a = amplitude of sinusoidal signal

END.